The Abel Lemma and the q-Gosper Algorithm

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Abstract

Chu has recently shown that the Abel lemma on summations by parts can serve as the underlying relation for Bailey's ${}_6\psi_6$ bilateral summation formula. In other words, the Abel lemma spells out the telescoping nature of the ${}_6\psi_6$ sum. We present a systematic approach to compute Abel pairs for bilateral and unilateral basic hypergeometric summation formulas by using the q-Gosper algorithm. It is demonstrated that Abel pairs can be derived from Gosper pairs. This approach applies to many classical summation formulas.

Keywords: the Abel lemma, Abel pairs, basic hypergeometric series, the q-Gosper algorithm, Gosper pairs.

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1 Introduction

We follow the notation and terminology in [10]. For $|q| < 1$, the q-shifted factorial is defined by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \text{ and } (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \text{ for } n \in \mathbb{Z}.$$

For convenience, we shall adopt the following notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n,$$

where n is an integer or infinity. In particular, for a nonnegative integer k, we have

$$(a;q)_{-k}=\frac{1}{(aq^{-k};q)_k}. \tag{1.1}$$

The (unilateral) basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left[\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix};q,z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_k}{(q,b_1,b_2,\ldots,b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k, \tag{1.2}$$

while the bilateral basic hypergeometric series ${}_r\psi_s$ is defined by

$${}_r\psi_s\left[\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix};q,z\right]=\sum_{k=-\infty}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_k}{(b_1,b_2,\ldots,b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{s-r} z^k. \tag{1.3}$$

Recently Chu [9] used the Abel lemma on summations by parts to give an elementary proof of Bailey's very well-poised ${}_6\psi_6$-series identity [5], see also, [10, Appendix II.33]:

$$\begin{aligned}&{}_6\psi_6\left[\begin{matrix} qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d, & e \\ a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & aq/e \end{matrix};q,\frac{qa^2}{bcde}\right] \\ &= \frac{(q,aq,q/a,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,qa^2/bcde;q)_\infty},\end{aligned} \tag{1.4}$$

where $|qa^2/bcde|<1$.

Let us give a brief review of Chu's approach. The Abel lemma on summation by parts is stated as

$$\sum_{k=-\infty}^{\infty}A_k(B_k-B_{k-1})=\sum_{k=-\infty}^{\infty}B_k(A_k-A_{k+1}) \tag{1.5}$$

provided that the series on both sides are convergent. Based on the Abel lemma, Chu found a pair (A_k,B_k):

$$A_k=\frac{(b,c,d,q^2a^2/bcd;q)_k}{(aq/b,aq/c,aq/d,bcd/aq;q)_k}, \tag{1.6}$$

and

$$B_k=\frac{(qe,bcd/a;q)_k}{(aq/e,q^2a^2/bcd;q)_k}\left(\frac{qa^2}{bcde}\right)^k, \tag{1.7}$$

which leads to the following iteration relation:

$$\begin{aligned}\Omega(a;b,c,d,e)&=\Omega(aq;b,c,d,eq)\\ &\times\frac{a(1-e)(1-aq)(1-aq/bc)(1-aq/bd)(1-aq/cd)}{e(1-a)(1-aq/b)(1-aq/c)(1-aq/d)(1-a^2q/bcde)},\end{aligned} \tag{1.8}$$

where

$$\Omega(a;b,c,d,e) = {}_6\psi_6 \left[\begin{matrix} qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d, & e \\ a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & aq/e \end{matrix} ; q, \frac{qa^2}{bcde} \right]. \tag{1.9}$$

Because of the symmetries in b, c, d, e, applying the identity (1.8) three times with respect to the parameters a and d, a and c, a and b, we arrive at the following iteration relation:

$$\begin{aligned} \Omega(a;b,c,d,e) &= \Omega(aq^4; bq, cq, dq, eq) \\ &\quad \times \frac{a^4q^6}{bcde} \cdot \frac{1-aq^4}{1-a} \cdot \frac{(1-b)(1-c)(1-d)(1-e)}{(a^2q/bcde;q)_4} \\ &\quad \times \frac{(aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de;q)_2}{(aq/b, aq/c, aq/d, aq/e;q)_3}. \end{aligned} \tag{1.10}$$

Again, iterating the above relation m times, we get

$$\begin{aligned} \Omega(a;b,c,d,e) &= \Omega(aq^{4m}; bq^m, cq^m, dq^m, eq^m) \\ &\quad \times \frac{a^{4m}q^{6m^2}}{(bcde)^m} \cdot \frac{1-aq^{4m}}{1-a} \cdot \frac{(b,c,d,e;q)_m}{(a^2q/bcde;q)_{4m}} \\ &\quad \times \frac{(aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de;q)_{2m}}{(aq/b, aq/c, aq/d, aq/e;q)_{3m}}. \end{aligned} \tag{1.11}$$

Replacing the summation index k with $k-2m$, we obtain the transformation formula

$$\begin{aligned} &\Omega(a;b,c,d,e) = \Omega(a; bq^{-m}, cq^{-m}, dq^{-m}, eq^{-m}) \\ &\quad \times \frac{(aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de;q)_{2m}}{(q/b, q/c, q/d, q/e, aq/b, aq/c, aq/d, aq/e;q)_m (a^2q/bcde;q)_{4m}}. \end{aligned} \tag{1.12}$$

Setting $m \to \infty$, Chu obtained the ${}_6\psi_6$ summation formula (1.4) by Jacobi's triple product identity [10, Appendix II.28]

$$\sum_{k=-\infty}^{\infty} q^{k^2} z^k = (q^2, -qz, -q/z; q^2)_\infty, \tag{1.13}$$

since

$$\begin{aligned} &\lim_{m\to\infty} \Omega(a; bq^{-m}, cq^{-m}, dq^{-m}, eq^{-m}) \\ &= \lim_{m\to\infty} {}_6\psi_6 \left[\begin{matrix} qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & bq^{-m}, & cq^{-m}, & dq^{-m}, & eq^{-m} \\ a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq^{m+1}/b, & aq^{m+1}/c, & aq^{m+1}/d, & aq^{m+1}/e \end{matrix} ; q, \frac{a^2q^{4m+1}}{bcde} \right] \\ &= \sum_{k=-\infty}^{\infty} \frac{1-aq^{2k}}{1-a} q^{2k^2-k} a^{2k} = \frac{1}{1-a} \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} a^k = (aq, q/a, q; q)_\infty. \end{aligned} \tag{1.14}$$

This paper is motivated by the question of how to systematically compute Abel pairs for bilateral summation formulas. We find that the q-Gosper is an efficient mechanism for this purpose. The q-Gosper algorithm has been extensively studied. Koorwinder gave a rigorous description of the q-Gosper algorithm in [14]. Abramov-Paule-Petkovšek [1] developed the algorithm `qHyper` for finding all q-hypergeometric solutions of linear homogeneous recurrences with polynomial coefficients. Later Böing-Koepf [7] gave an algorithm for the same purpose. The Maple package `qsum.mpl` was described by Böing-Koepf [7]. In [16], Riese presented an generalization of the q-Gosper's algorithm to indefinite bibasic hypergeometric summations.

Recall that a function t_k is called a basic hypergeometric term if t_{k+1}/t_k is a rational function of q^k. The q-Gosper algorithm is devised to answer the question if there is a basic hypergeometric term z_k for a given basic hypergeometric term t_k such that

$$t_k = z_{k+1} - z_k. \tag{1.15}$$

We observe that for an iteration relation of a summation formula, the difference of the kth term of the both sides is a basic hypergeometric term for which the q-Gosper algorithm can be employed.

The main result of this paper is to present a general framework to deal with basic hypergeometric identities based on the q-Gosper algorithm. We always start with an iteration relation. Then we use the q-Gosper algorithm to generate a Gosper pair (g_k, h_k) if it exists. We next turn to the iteration relation and derive the desired identity by computing the limit value. Actually, once aGosper pair (g_k, h_k) is obtained, one can easily compute the corresponding Abel pair. Indeed, the Abel pair for the ${}_6\psi_6$ sum discovered by Chu [9] agrees with the Abel pair derived from the Gopser pair by using our approach. In general, our method is efficient for many classical summation formulas with parameters.

As examples, we give Gosper pairs and Abel pairs of several well-known bilateral summation formulas including Ramanujan's ${}_1\psi_1$ summation formula. In the last section we demonstrate that the idea of Gosper pairs can be applied to unilateral summation formulas as well. We use the q-Gauss ${}_2\phi_1$ summation formula as an example to illustrate the procedure to compute Gosper pairs. As another example, we derive the Gosper pair and the Abel pair of the ${}_6\phi_5$ summation formula.

In comparison with a recent approach presented by Chen-Hou-Mu [8] for proving nonterminating basic hypergeometric series identities by using the q-Zeilberger algorithm, one sees that the approach we undertake in this paper does not rely on the introduction of the parameter n in order to establish recurrence relations, and only makes use of the q-Gosper algorithm.

2 The Gosper Pairs for Bilateral Summations

In spite of its innocent looking, the Abel lemma is intrinsic for some sophisticated bilateral basic hypergeometric identities. In this section, we introduce the notion of Gosper pairs and show that one may apply the q-Gosper algorithm to construct Gosper pairs which can be regarded as certificates like the Abel pairs to justify iteration relations for bilateral summations. Furthermore, it is easily seen that one can compute the Abel pairs from Gosper pairs.

Suppose that we have a bilateral series $\sum_{-\infty}^{\infty} F_k(a_1, a_2, \ldots, a_n)$ which has a closed form. Making the substitutions $a_i \to a_i q$ or $a_i \to a_i/q$ for some parameters a_i, the closed product formula induces an iteration relation for the summation which can be stated as an identity of the form:

$$\sum_{k=-\infty}^{\infty} F_k(a_1, a_2, \ldots, a_n) = \sum_{k=-\infty}^{\infty} G_k(a_1, a_2, \ldots, a_n). \tag{2.1}$$

We assume that $\sum_{k=-\infty}^{\infty} F_k(a_1, a_2, \ldots, a_n)$ and $\sum_{k=-\infty}^{\infty} G_k(a_1, a_2, \ldots, a_n)$ are both convergent. We also assume that

$$\lim_{k\to\infty} h_k = \lim_{k\to-\infty} h_k. \tag{2.2}$$

We note that there are many bilateral summations with the above limit property.

An Gosper pair (g_k, h_k) is a pair of basic hypergeometric terms such that

$$\begin{aligned} g_k - h_k &= F_k(a_1, a_2, \ldots, a_n), \\ g_k - h_{k+1} &= G_k(a_1, a_2, \ldots, a_n). \end{aligned}$$

Evidently, once a Gosper pair is derived, the identity (2.1) immediately justified by the following telescoping relation:

$$\sum_{k=-\infty}^{\infty} (g_k - h_k) = \sum_{k=-\infty}^{\infty} (g_k - h_{k+1}). \tag{2.3}$$

We are now ready to describe our approach. Let us take Ramanujan's ${}_1\psi_1$ sum [10, Appendix II.29] as an example:

$${}_1\psi_1 \left[\begin{matrix} a \\ b \end{matrix} ; q, z \right] = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \tag{2.4}$$

where $|b/a| < |z| < 1$. There are many proofs of this identity, see, for example, Hahn [11], Jackson [13], Andrews [2,3], Ismail [12], Andrews and Askey [4], and Berndt [6].

Proposition 2.1 *The following is a Gosper pair for Ramanujan's* ${}_1\psi_1$ *sum:*

$$g_k = \frac{az^{k+1}}{(az-b)}\frac{(a;q)_k}{(b;q)_k},$$

$$h_k = \frac{bz^k}{(az-b)}\frac{(a;q)_k}{(b;q)_k}.$$

Step 1. Construct an iteration relation from the closed product form, namely, the right hand side of (2.4).

Setting b to bq in (2.4), we get

$$ {}_1\psi_1 \left[\begin{array}{c} a \\ bq \end{array} ; q, z \right] = \frac{(q, bq/a, az, q/az; q)_\infty}{(bq, q/a, z, bq/az; q)_\infty}. \tag{2.5}$$

Define

$$f(a,b,z) = {}_1\psi_1 \left[\begin{array}{c} a \\ b \end{array} ; q, z \right]. \tag{2.6}$$

Comparing the right hand sides of (2.4) and (2.5) gives the following iteration relation (see also [4])

$$ {}_1\psi_1 \left[\begin{array}{c} a \\ b \end{array} ; q, z \right] = \frac{(1-b/a)}{(1-b)(1-b/az)}\ {}_1\psi_1 \left[\begin{array}{c} a \\ bq \end{array} ; q, z \right]. \tag{2.7}$$

Notice that both sides of the above identity are convergent.

Let $F_k(a,b,z)$ and $G_k(a,b,z)$ denote the kth terms of the left hand side and the right hand side summations in (2.7) respectively, that is,

$$F_k(a,b,z) = \frac{(a;q)_k}{(b;q)_k} z^k \quad \text{and} \quad G_k(a,b,z) = \frac{(1-b/a)}{(1-b)(1-b/az)} F_k(a,bq,z). \tag{2.8}$$

Step 2. Apply the q-Gosper algorithm to try to find a Gosper pair (g_k, h_k).

It is essential to observe that $F_k(a,b,z) - G_k(a,b,z)$ is a basic hypergeometric term. In fact it can be written as

$$\left(1 - bq^k - \frac{1-b/a}{1-b/az}\right) \frac{(a;q)_k}{(b;q)_{k+1}} z^k.$$

Now we may employ the q-Gosper algorithm for the following equation

$$F_k(a,b,z) - G_k(a,b,z) = h_{k+1} - h_k, \tag{2.9}$$

and we find a solution of simple form

$$h_k = \frac{bz^k}{(az-b)}\frac{(a;q)_k}{(b;q)_k}, \tag{2.10}$$

which also satisfied the limit condition

$$\lim_{k\to\infty} h_k = \lim_{k\to-\infty} h_k = 0.$$

As far as the verification of (2.7) is concerned, the existence of a solution h_k and the limit condition (2.2) would guarantee that the identity holds. Now it takes one more step to compute the Gosper pair:

$$g_k = h_k + F_k(a,b,z) = \frac{az^{k+1}}{(az-b)}\frac{(a;q)_k}{(b;q)_k}. \tag{2.11}$$

Step 3. Based on the iteration relation and the limit values, one can verify the summation formula.

From the iteration relation (2.7), we may reduce the evaluation of the bilateral series ${}_1\psi_1$ to a special case

$$f(a,b,z) = \frac{(b/a;q)_\infty}{(b,b/za;q)_\infty}\, f(a,0,z). \tag{2.12}$$

Setting $b=q$ in (2.12), we get

$$f(a,0,z) = \frac{(q,q/za;q)_\infty}{(q/a;q)_\infty}\sum_{k=-\infty}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k.$$

Invoking the relation (1.1), we see that $1/(q,q)_{-k} = 0$ for any positive integer k. Consequently, the above bilateral sum reduces to a unilateral sum. Exploiting the q-binomial theorem [10, Appendix II.3]

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{2.13}$$

we get the evaluation

$$f(a,0,z) = \frac{(q,az,q/az;q)_\infty}{(q/a,z;q)_\infty}. \tag{2.14}$$

Hence the identity (2.4) follows from (2.12) and (2.14).

It should be warned that it is not always the case that there is a solution h_k to the equation (2.9) in general case. If one encounters this scenario, one should still have alternatives to try another iteration relations, as is done for the ${}_3\psi_3$ sum in Example 2.4.

Let us now examine how to generate an Abel pair (A_k, B_k) from a Gosper pair (g_k, h_k). Setting

$$g_k = A_k B_k \quad\text{and}\quad h_k = A_k B_{k-1}, \tag{2.15}$$

then we see that

$$\frac{B_k}{B_{k-1}} = \frac{g_k}{h_k}. \tag{2.16}$$

Without loss of generality, we may assume that $B_0 = 1$. Iterating (2.16) yields an Abel pair (A_k, B_k).

For the Ramanujan's ${}_1\psi_1$ sum (2.4), we can compute the Abel pair by using the q-Gosper algorithm.

Proposition 2.2 *The following is an Abel pair for Ramanujan's ${}_1\psi_1$ sum:*

$$\begin{aligned} A_k &= \frac{az}{az-b}\frac{(a;q)_k}{(b;q)_k}\left(\frac{b}{a}\right)^k, \\ B_k &= \left(\frac{az}{b}\right)^k. \end{aligned}$$

It is a routine to verify (A_k, B_k) is indeed an Abel pair for the ${}_1\psi_1$ sum. First we have

$$B_k - B_{k-1} = \left(\frac{az}{b}-1\right)\left(\frac{az}{b}\right)^{k-1}, \tag{2.17}$$

$$A_k - A_{k+1} = \frac{(1-b/a)}{(1-b)(1-b/az)}\frac{(a;q)_k}{(bq;q)_k}\left(\frac{b}{a}\right)^k. \tag{2.18}$$

Then the iteration relation (2.7) is deduced from the Abel lemma:

$$\sum_{k=-\infty}^{\infty} \frac{az}{az-b}\frac{(a;q)_k}{(b;q)_k}\left(\frac{b}{a}\right)^k\left(\frac{az}{b}-1\right)\left(\frac{az}{b}\right)^{k-1} \tag{2.19}$$

$$= \sum_{k=-\infty}^{\infty}\left(\frac{az}{b}\right)^k\frac{(1-b/a)}{(1-b)(1-b/az)}\frac{(a;q)_k}{(bq;q)_k}\left(\frac{b}{a}\right)^k. \tag{2.20}$$

We next give some examples for bilateral summations.

Example 2.3 The sum of a well-poised ${}_2\psi_2$ series ([10], Appendix II.30):

$${}_2\psi_2\left[\begin{matrix} b, & c \\ aq/b, & aq/c\end{matrix}; q, -\frac{aq}{bc}\right] = \frac{(aq/bc;q)_\infty(aq^2/b^2, aq^2/c^2, q^2, aq, q/a; q^2)_\infty}{(aq/b, aq/c, q/b, q/c, -aq/bc; q)_\infty}, \tag{2.21}$$

where $|aq/bc| < 1$.

Write the kth term of the left hand side of (2.21) as

$$F_k(a,b,c) = \frac{(b,c;q)_k}{(aq/b, aq/c;q)_k}\left(-\frac{aq}{bc}\right)^k. \tag{2.22}$$

Substituting b with b/q in (2.21), we are led to the iteration relation

$$\sum_{k=-\infty}^{\infty} F_k(a,b,c) = \frac{(1-aq/bc)(1-aq^2/b^2)}{(1+aq/bc)(1-q/b)(1-aq/b)} \sum_{k=-\infty}^{\infty} F_k(a,b/q,c). \tag{2.23}$$

Let

$$G_k(a,b,c) = \frac{(1-aq/bc)(1-aq^2/b^2)}{(1+aq/bc)(1-q/b)(1-aq/b)} F_k(a,b/q,c). \tag{2.24}$$

Implementing the q-Gosper algorithm, we obtain a Gosper pair

$$g_k = \frac{(b^2cq^k-aq^2)}{(aq+bc)(bq^k-q)} F_k(a,b,c), \tag{2.25}$$

$$h_k = \frac{bq(c-aq^k)}{(aq+bc)(bq^k-q)} F_k(a,b,c). \tag{2.26}$$

The companion Abel pair is given below:

$$A_k = \frac{(b,c;q)_k(b^2cq^k-aq^2)}{(aq/b,b^2c/aq;q)_k(aq+bc)(-q+bq^k)}, \tag{2.27}$$

$$B_k = \frac{(b^2c/aq;q)_k}{(aq/c;q)_k}\left(-\frac{aq}{bc}\right)^k. \tag{2.28}$$

Noticing that (2.21) is symmetric in b and c, we have

$$\sum_{k=-\infty}^{\infty} F_k(a,b,c) = \frac{(1-aq/bc)(1-aq^2/bc)(1-aq^2/b^2)(1-aq^2/c^2)}{(1+aq/bc)(1+aq^2/bc)(1-q/b)(1-q/c)(1-aq/b)(1-aq/c)} \times \sum_{k=-\infty}^{\infty} F_k(a,b/q,c/q). \tag{2.29}$$

Finally, we can reach (2.21) by iterating (2.29) infinitely many times along with Jacobi's triple product identity (1.13) as the limit case.

Example 2.4 Bailey's sum of a well-poised ${}_3\psi_3$ ([10], Appendix II.31):

$${}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix}; q, \frac{q}{bcd}\right] = \frac{(q,q/bc,q/bd,q/cd;q)_\infty}{(q/b,q/c,q/d,q/bcd;q)_\infty}, \tag{2.30}$$

where $|q/bcd|<1$.

Substituting d with d/q in (2.30), one obtains the iteration relation

$${}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix}; q, \frac{q}{bcd}\right] = \frac{(1-q/bd)(1-q/cd)}{(1-q/d)(1-q/bcd)}\, {}_3\psi_3\left[\begin{matrix} q, & c, & d/q \\ q/b, & q/c, & q^2/d \end{matrix}; q, \frac{q^2}{bcd}\right]. \tag{2.31}$$

We remark that this sum is in fact an example for which the q-Gosper algorithm does not succeed for the iteration relation derived from a straightforward substitution such as $d \to dq$ or $d \to d/q$. Instead, using an idea of Paule [15] of symmetrizing a bilateral summation, we replace k by $-k$ on the left hand side of (2.30) to get

$$ {}_3\psi_3 \left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix} ; q, \frac{q^2}{bcd} \right]. \tag{2.32} $$

Let $F_k(b,c,d)$ be the average of the kth summands of (2.30) and (2.32), namely,

$$ F_k(b,c,d) = \frac{(b,c,d;q)_k}{(q/b,q/c,q/d;q)_k} \left(\frac{q}{bcd}\right)^k \frac{1+q^k}{2}, \tag{2.33} $$

and let

$$ G_k(b,c,d) = \frac{(1-q/bd)(1-q/cd)}{(1-q/d)(1-q/bcd)} F_k(b,c,d/q). \tag{2.34} $$

With regard to $F_k(b,c,d) - G_k(b,c,d)$, the q-Gosper algorithm generates a Gosper pair:

$$ \begin{aligned} g_k &= \frac{bdq^{k+1} + cdq^{k+1} - bcd^2q^k - q^2 + dq^{k+1} + bcdq^{k+1} - bcd^2q^{2k} - q^{k+2}}{(1+q^k)(bcd-q)(q-dq^k)} \\ &\quad \times F_k(b,c,d), \end{aligned} \tag{2.35} $$

$$ h_k = \frac{d(b-q^k)(c-q^k)}{(1+q^k)(q-bcd)(1-dq^{k-1})} F_k(b,c,d), \tag{2.36} $$

which implies the iteration relation (2.31). Invoking the symmetric property of the parameters b, c and d, we have

$$ \begin{aligned} {}_3\psi_3 \left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix} ; q, \frac{q}{bcd} \right] &= \frac{(1-q/bc)(1-q^2/bc)(1-q/bd)(1-q^2/bd)}{(1-q/b)(1-q/c)(1-q/d)(1-q/bcd)} \\ &\quad \times \frac{(1-q/cd)(1-q^2/cd)}{(1-q^2/bcd)(1-q^3/bcd)} \, {}_3\psi_3 \left[\begin{matrix} b/q, & c/q, & d/q \\ q^2/b, & q^2/c, & q^2/d \end{matrix} ; q, \frac{q^4}{bcd} \right]. \end{aligned} \tag{2.37} $$

The above relation enables us to reduce the summation formula (2.30) to Jacobi's triple product identity.

Example 2.5 A basic bilateral analogue of Dixon's sum [10, Appendix II.32]:

$$ \begin{aligned} &{}_4\psi_4 \left[\begin{matrix} -qa^{\frac{1}{2}}, & b, & c, & d \\ -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d \end{matrix} ; q, \frac{qa^{\frac{3}{2}}}{bcd} \right] \\ &= \frac{(aq, aq/bc, aq/bd, aq/cd, qa^{\frac{1}{2}}/b, qa^{\frac{1}{2}}/c, qa^{\frac{1}{2}}/d, q, qa; q)_\infty}{(aq/b, aq/c, aq/d, q/b, q/c, q/d, qa^{\frac{1}{2}}, qa^{-\frac{1}{2}}, qa^{\frac{3}{2}}/bcd; q)_\infty}, \end{aligned} \tag{2.38} $$

where $|qa^{\frac{3}{2}}/bcd| < 1$.

For the above formula, we may consider the substitution $d \to d/q$ in (2.38) which suggests the iteration relation

$$
\begin{aligned}
&{}_4\psi_4\left[\begin{matrix} -qa^{\frac{1}{2}}, & b, & c, & d \\ -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d \end{matrix} ; q, \frac{qa^{\frac{3}{2}}}{bcd}\right] \\
&= \frac{(1-aq/bd)(1-aq/cd)(1-qa^{\frac{1}{2}}/d)}{(1-aq/d)(1-q/d)(1-qa^{\frac{3}{2}}/bcd)} \\
&\quad \times {}_4\psi_4\left[\begin{matrix} -qa^{\frac{1}{2}}, & b, & c, & d/q \\ -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq^2/d \end{matrix} ; q, \frac{q^2a^{\frac{3}{2}}}{bcd}\right].
\end{aligned} \tag{2.39}
$$

Let

$$
F_k(a,b,c,d) = \frac{(-qa^{\frac{1}{2}}, b, c, d; q)_k}{(-a^{\frac{1}{2}}, aq/b, aq/c, aq/d)} \left(\frac{qa^{\frac{3}{2}}}{bcd}\right)^k \tag{2.40}
$$

and let

$$
G_k(a,b,c,d) = \frac{(1-aq/bd)(1-aq/cd)(1-qa^{\frac{1}{2}}/d)}{(1-aq/d)(1-q/d)(1-qa^{\frac{3}{2}}/bcd)} F_k(a,b,c,d/q). \tag{2.41}
$$

By computation we obtain the Gosper pair:

$$
\begin{aligned}
g_k &= \frac{-abdq^{k+1} - acdq^{k+1} + q^2a^{\frac{3}{2}} + a^2q^{k+2} - bcda^{\frac{1}{2}}q^{k+1} - da^{\frac{3}{2}}q^{k+1} + bcd^2q^k + bcd^2a^{\frac{1}{2}}q^{2k}}{(dq^k - q)(1 + a^{\frac{1}{2}}q^k)(bcd - a^{\frac{3}{2}}q)} \\
&\quad \times F_k(a,b,c,d),
\end{aligned} \tag{2.42}
$$

$$
h_k = \frac{d(aq^k - c)(aq^k - b)}{(dq^{k-1} - 1)(1 + a^{\frac{1}{2}}q^k)(bcd - qa^{\frac{3}{2}})} F_k(a,b,c,d). \tag{2.43}
$$

So the iteration relation (2.39) holds. From the symmetric property of the parameters b, c and d, we have

$$
\begin{aligned}
&{}_4\psi_4\left[\begin{matrix} -qa^{\frac{1}{2}}, & b, & c, & d \\ -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d \end{matrix} ; q, \frac{qa^{\frac{3}{2}}}{bcd}\right] \\
&= \frac{(1-aq/bc)(1-aq^2/bc)(1-aq/bd)(1-aq^2/bd)(1-aq/cd)(1-aq^2/cd)}{(1-aq/b)(1-aq/c)(1-aq/d)(1-q/b)(1-q/c)(1-q/d)(1-qa^{\frac{3}{2}}/bcd)} \\
&\quad \times \frac{(1-qa^{\frac{1}{2}}/b)(1-qa^{\frac{1}{2}}/c)(1-qa^{\frac{1}{2}}/d)}{(1-q^2a^{\frac{3}{2}}/bcd)(1-q^3a^{\frac{3}{2}}/bcd)} \\
&\quad \times {}_4\psi_4\left[\begin{matrix} -qa^{\frac{1}{2}}, & b/q, & c/q, & d/q \\ -a^{\frac{1}{2}}, & aq^2/b, & aq^2/c, & aq^2/d \end{matrix} ; q, \frac{q^4a^{\frac{3}{2}}}{bcd}\right].
\end{aligned} \tag{2.44}
$$

By iteration, it follows that

$$
\begin{aligned}
&{}_4\psi_4\left[\begin{matrix} -qa^{\frac{1}{2}}, & b, & c, & d \\ -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d \end{matrix}; q, \frac{qa^{\frac{3}{2}}}{bcd}\right] \\
&= \frac{(aq/bc, aq/bd, aq/cd, qa^{\frac{1}{2}}/b, qa^{\frac{1}{2}}/c, qa^{\frac{1}{2}}/d; q)_\infty}{(aq/b, aq/c, aq/d, q/b, q/c, q/d, qa^{\frac{3}{2}}/bcd; q)_\infty} H(a), \qquad (2.45)
\end{aligned}
$$

where

$$
H(a) = \sum_{k=-\infty}^{\infty} \frac{(-qa^{\frac{1}{2}}; q)_k}{(-a^{\frac{1}{2}}; q)_k} q^{3\binom{k}{2}} \left(-qa^{\frac{3}{2}}\right)^k. \qquad (2.46)
$$

Taking $b = -a^{\frac{1}{2}}$ and $c, d \to \infty$ in (2.45) and by Jacobi's triple product identity (1.13), it can be verified that

$$
H(a) = \frac{(q, aq, q/a; q)_\infty}{(qa^{\frac{1}{2}}, qa^{-\frac{1}{2}})_\infty}, \qquad (2.47)
$$

which leads to (2.38).

Example 2.6 Bailey's very well-poised ${}_6\psi_6$-series identity (1.4).

Let us denote the kth term of (1.9) as

$$
\Omega_k(a; b, c, d, e) = \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; q)_k}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q)_k} \left(\frac{qa^2}{bcde}\right)^k. \qquad (2.48)
$$

Set $F_k = \Omega_k(a; b, c, d, e)$ and

$$
G_k = \Omega_k(aq; b, c, d, eq) \times \frac{a(1-e)(1-aq)(bc-aq)(bd-aq)(cd-aq)}{(1-a)(b-aq)(c-aq)(d-aq)(bcde-a^2q)}. \qquad (2.49)
$$

By computation, we find the following Gosper pair:

$$
\begin{aligned}
g_k &= \frac{a(bcdq^k - aq)(1-eq^k)}{(bcde-a^2q)(1-aq^{2k})} \Omega_k(a; b, c, d, e), \\
h_k &= \frac{(e-aq^k)(bcd-a^2q^{k+1})}{(bcde-a^2q)(aq^{2k}-1)} \Omega_k(a; b, c, d, e). \qquad (2.50)
\end{aligned}
$$

We note that the Abel pair derived from the above Gosper pair coincides with the Abel pair given by Chu [9].

We next give the Gosper pair of a different iteration relation of the ${}_6\psi_6$ series by setting $e \to e/q$ in (1.4), that is,

$$
\Omega(a; b, c, d, e) = \Omega(a; b, c, d, e/q) \times \frac{(1-aq/be)(1-aq/ce)(1-aq/de)}{(1-aq/e)(1-q/e)(1-a^2q/bcde)}. \qquad (2.51)
$$

We will see that the above iteration has the advantage that it directly leads to the identity (1.12) by taking into account the symmetries in b, c, d, e. On the other hand, in this case the Gosper pair does not have a simple expression.

Set $F_k = \Omega_k(a; b, c, d, e)$ and

$$G_k = \Omega_k(a; b, c, d, e/q) \times \frac{(1-aq/be)(1-aq/ce)(1-aq/de)}{(1-aq/e)(1-q/e)(1-a^2q/bcde)}. \tag{2.52}$$

From the above iteration relation, we obtain the Gosper pair:

$$\begin{aligned} g_k &= \left(\frac{abceq^{k+1} + abdeq^{k+1} - a^2beq^{2k+1} + acdeq^{k+1} - a^2ceq^{2k+1} - a^2deq^{2k+1} - bcde^2q^k}{(bcde - qa^2)(eq^k - q)(aq^{2k} - 1)} \right. \\ &\quad \left. + \frac{abcde^2q^{3k} - abcdeq^{2k+1} - a^2q^2 + a^2eq^{k+1} + a^3q^{2k+2}}{(bcde - qa^2)(eq^k - q)(aq^{2k} - 1)} \right) \Omega_k(a; b, c, d, e), \qquad (2.53) \\ h_k &= \frac{qe(b - aq^k)(c - aq^k)(d - aq^k)}{(bcde - qa^2)(1 - aq^{2k})(eq^k - q)} \Omega_k(a; b, c, d, e). \qquad (2.54) \end{aligned}$$

Since the parameters b, c, d, e are symmetric in (1.9), we obtain

$$\begin{aligned} \Omega(a; b, c, d, e) = \Omega(a; b/q, c/q, d/q, e/q) \\ &\times \frac{(1-aq/bc)(1-aq^2/bc)(1-aq/bd)(1-aq^2/bd)}{(1-aq/b)(1-aq/c)(1-aq/d)(1-aq/e)} \\ &\times \frac{(1-aq/be)(1-aq^2/be)(1-aq/cd)(1-aq^2/cd)}{(1-q/b)(1-q/c)(1-q/d)(1-q/e)} \\ &\times \frac{(1-aq/ce)(1-aq^2/ce)(1-aq/de)(1-aq^2/de)}{(1-a^2q/bcde)(1-a^2q^2/bcde)(1-a^2q^3/bcde)(1-a^2q^4/bcde)}. \end{aligned} \tag{2.55}$$

Again, the limit value can be given by Jacobi's triple product identity, so that we arrive at (1.4) in view of (2.55).

To conclude this section, we remark that our approach is feasible for the computation of a Gosper pair for the ${}_{10}\psi_{10}$ summation formula of Chu [9].

3 The Abel Pairs for Unilateral Summations

The idea of Gosper pairs can be adapted to unilateral summation formulas with a slight modification. We also begin with an iteration relation guided by the closed product formula which can be stated in the following form:

$$\sum_{k=0}^{\infty} F_k(a_1, a_2, \ldots, a_n) = \sum_{k=0}^{\infty} G_k(a_1, a_2, \ldots, a_n). \tag{3.1}$$

We assume that $\sum_{k=0}^{\infty} F_k(a_1, a_2, \ldots, a_n)$ and $\sum_{k=0}^{\infty} G_k(a_1, a_2, \ldots, a_n)$ are convergent. Moreover, we assume that the following limit condition holds:

$$\lim_{k\to\infty} h_k = h_0. \tag{3.2}$$

For the same reason as in the bilateral case, we see that

$$F_k(a_1, a_2, \ldots, a_n) - G_k(a_1, a_2, \ldots, a_n)$$

is a basic hypergeometric terms so that we can resort to the q-Gosper algorithm to solve the following equation:

$$F_k(a_1, a_2, \ldots, a_n) - G_k(a_1, a_2, \ldots, a_n) = h_{k+1} - h_k, \quad k \geq 0. \tag{3.3}$$

A Gosper pair (g_k, h_k) is then given by

$$\begin{aligned} g_k - h_k &= F_k(a_1, a_2, \ldots, a_n), \\ g_k - h_{k+1} &= G_k(a_1, a_2, \ldots, a_n). \end{aligned}$$

Therefore, one can use the Gosper pair (g_k, h_k) to justify (3.1) by the relation

$$\sum_{k=0}^{\infty}(g_k - h_k) = \sum_{k=0}^{\infty}(g_k - h_{k+1}) \tag{3.4}$$

and the limit condition $\lim\limits_{k\to\infty} h_k = h_0$.

Given a Gosper pair it is easy to compute the corresponding Abel pair which implies iteration relation (3.1) by the following unilateral Abel sum:

$$\sum_{k=0}^{\infty} A_k(B_k - B_{k-1}) = \sum_{k=0}^{\infty} B_k(A_k - A_{k+1}), \tag{3.5}$$

which we call the unilateral Abel lemma.

The above approach is applicable to many classical unilateral summation formulas including the q-Gauss sum, the q-Kummer (Bailey-Daum) sum [10, Appendix II.9], the q-Dixon sum [10, Appendix II.13], a q-analogue of Watson's ${}_3F_2$ sum [10, Appendix II.16], and a q-analogue of Whipple's ${}_3F_2$ sum [10, Appendix II.18], just to name a few. Here we only give two examples to demonstrate this technique.

Example 3.1 The q-Gauss sum:

$${}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} ; q, \frac{c}{ab} \right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \tag{3.6}$$

where $|c/ab| < 1$.

Set

$$f(a,b,c) = {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ; q, \frac{c}{ab}\right]. \tag{3.7}$$

Write the kth term of (3.7) as

$$F_k(a,b,c) = \frac{(a,b;q)_k}{(q,c;q)_k}\left(\frac{c}{ab}\right)^k. \tag{3.8}$$

The iteration $c \to cq$ in (3.6) implies

$$f(a,b,c) = \frac{(1-c/a)(1-c/b)}{(1-c)(1-c/ab)} f(a,b,cq). \tag{3.9}$$

Let

$$G_k(a,b,c) = \frac{(1-c/a)(1-c/b)}{(1-c)(1-c/ab)} F_k(a,b,cq). \tag{3.10}$$

Applying the q-Gosper algorithm to $F_k(a,b,c) - G_k(a,b,c)$, we arrive at the Gosper pair:

$$g_k = \frac{c-abq^k}{c-ab} F_k(a,b,c), \tag{3.11}$$

$$h_k = \frac{ab(1-q^k)}{c-ab} F_k(a,b,c). \tag{3.12}$$

So we have the Abel pair:

$$A_k = \frac{(1-abq^k/c)}{(1-ab/c)} \frac{(a,b;q)_k}{(c,abq/c;q)_k}, \tag{3.13}$$

$$B_k = \frac{(abq/c;q)_k}{(q;q)_k}\left(\frac{c}{ab}\right)^k. \tag{3.14}$$

Now we see that identity (3.6) is true because of the unilateral Abel lemma (3.5) and the limit value $f(a,b,0) = 1$.

Example 3.2 The sum of Rogers' nonterminating very-well-poised ${}_6\phi_5$ series [10, Appendix II.20]:

$$\begin{aligned} &{}_6\phi_5\left[\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d \end{matrix} ; q, \frac{aq}{bcd}\right] \\ &\qquad = \frac{(aq, aq/bc, aq/bd, aq/cd;q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd;q)_\infty}, \end{aligned} \tag{3.15}$$

where $|aq/bcd| < 1$.

Let us write

$$f(a,b,c,d) = {}_6\phi_5 \left[\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & d \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d \end{matrix} ; q, \frac{aq}{bcd} \right]. \tag{3.16}$$

Denote the kth term of (3.16) by

$$F_k(a,b,c,d) = \frac{(a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d; q)_k}{(q, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d; q)_k} \left(\frac{aq}{bcd}\right)^k. \tag{3.17}$$

The substitution $a \to aq$ in (3.15) leads to the iteration relation

$$f(a,b,c,d) = \frac{(1-aq)(1-aq/cd)(1-aq/bc)(1-aq/bd)}{(1-aq/b)(1-aq/c)(1-aq/d)(1-aq/bcd)} f(aq,b,c,d). \tag{3.18}$$

Let

$$G_k(a,b,c,d) = \frac{(1-aq)(1-aq/cd)(1-aq/bc)(1-aq/bd)}{(1-aq/b)(1-aq/c)(1-aq/d)(1-aq/bcd)} F_k(aq,b,c,d). \tag{3.19}$$

By computation we get the Gosper pair:

$$g_k = \frac{(1-aq^k)(q^k - aq/bcd)}{(1-aq^{2k})(1-aq/bcd)} F_k(a,b,c,d), \tag{3.20}$$

$$h_k = -\frac{(1-q^k)(1-a^2q^{k+1}/bcd)}{(1-aq^{2k})(1-aq/bcd)} F_k(a,b,c,d). \tag{3.21}$$

The corresponding Abel pair is as follows:

$$A_k = \frac{(bcdq^k - aq)}{(bcd - aq)} \frac{(b, c, d, a^2q^2/bcd; q)_k}{(aq/b, aq/c, aq/d, bcd/a; q)_k}, \tag{3.22}$$

$$B_k = \frac{(aq, bcd/a; q)_k}{(q, a^2q^2/bcd; q)_k} \left(\frac{aq}{bcd}\right)^k. \tag{3.23}$$

Therefore, the identity (3.15) is a consequence of the unilateral Abel lemma (3.5) and the limit value $f(0,b,c,d) = 1$.

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